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ESTIMATING PURITY AND ENTROPY IN STABILIZER STATE EXPERIMENTS

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Many experiments in quantum information aim at creating graph states. Quantifying the purity of an experimentally achieved graph state could in principle be accomplished using full-state tomography. This method requires a number of measurement settings growing exponentially with the number of constituents involved. Thus, full-state tomography becomes experimentally infeasible even for a moderate number of qubits.

In this paper we present a method to estimate the purity of experimentally achieved graph states with simple measurements. The observables we consider are the stabilizers of the underlying graph. Then, we formulate the problem as: "What is the state with the least purity that is compatible with the measurement data?" We solve this problem analytically and compare the obtained bounds with results from full-state tomography for simulated data.

Keywords: Graph state; Stabilizer; Purity; Distributed QIP.

1. Introduction

Stabilizer states [1] (notably including cluster states [2]) represent a major class of entangled states. They form the resource for a number of applications in quantum information, such as quantum computing[3], quantum error correction[4], and quantum cryptography[5]. Because of the usefulness of cluster states, a considerable effort has been devoted to their theoretical studies as well as to their experimental implementation. To date, photonic cluster states of four[6] and six qubits[7] have been experimentally demonstrated, and their realization with trapped ions is actively pursued[8].

The experimental progress made in the area of cluster states effects the need for sophisticated methods for the estimation of the system's properties. A natural problem which arises is the fact that the density matrix grows exponentially with the number of qubits involved. So far, it has already been shown that fidelities and entanglement in cluster state experiments can be estimated using a number of observables linear in the number of qubits[9,10]. The observables of choice are the so-called stabilizers (see e.g. Ref. 11 for an introduction to the stabilizer formalism), which are given by

$$K_j = X_j Z_{N_j}, \tag{1}$$

where X denotes the usual Pauli x operator acting on qubit j, and Z_{N_j} denotes the Pauli z operator acting on neighbours of j defined by the underlying graph. Note that for a graph of n qubits, the set $\{K_1, \dots, K_n\}$ generates an abelian group, called the stabilizer group. Graph states form the simultaneous eigenvector with eigenvalue $+1$ to these stabilizers.

In this paper we will show how to estimate the purity $tr(\rho^2)$ in cluster state experiments. We will consider the generators of the stabilizer group as the observables. Similar to the estimation of entanglement in Refs. 12 and 13 (cfr. Refs. 14 and 15 for a review of entanglement estimation methods), the problem is formulated as: "What is the quantum state with the lowest purity, and which is compatible with the measurement data?" Mathematically, this question is the following quadratic optimization problem:

$$P_{min} = min_\rho \left[tr(\rho^2) : tr(K_i \rho) = a_i, \rho \geq 0 \right]. \tag{2}$$

The paper is structured as follows: in Sec. 2.1 we will utilize the symmetries allowed by the observables to restrict the optimization to stabilizer diagonal states. Then, we provide an exact analytical solution to the above optimization problem in Sec. 2.2. The following section proceeds with a discussion of the quality of the obtained solution by comparing it with results from full-state tomography from simulated data of a noisy system. Sec. 6 discusses the estimation of the von-Neumann entropy from stabilizer measurements, and provides an exact solution to the maximal entropy problem.

2. Estimating purities

2.1. *Symmetries of the stabilizers*

As mentioned in the introduction, the natural observables in cluster state experiments are given by the stabilizers. Let us assume the goal of an experiment was the creation of a pure cluster state, with stabilizers K_j, $j \in \{1, \dots, n\}$. We denote the measurement outcomes by $a_j = tr(\rho K_j)$. Note that these expectation values are invariant under rotations of the stabilizer group. By twirling over the stabilizer, one may therefore restrict to states of the form[17]

$$\rho = \frac{1}{2^n} \sum_{i_1,\ldots,\imath_n=0}^{1} c_{i_1\ldots i_n} K_1^{i_1} \ldots K_n^{i_n} . \tag{3}$$

Here, the following twirling protocol was utilized:

$$\rho' \longrightarrow \rho = \frac{1}{2^n} \sum_{i_1,\ldots,i_n=0}^{1} K_1^{i_1} \ldots K_n^{i_n} \rho' K_1^{i_1} \ldots K_n^{i_n} . \tag{4}$$

The coefficients in the stabilizer decomposition of ρ are subject to the normalization constraint $c_{0\ldots0} = 1$ and measurement outcomes $a_j = tr(\rho K_j) = c_{\tilde{j}}$, where $\tilde{j}$ denotes a bit string of zeros, but 1 in position j. Furthermore, any valid density matrix must be positive-semidefinite. The stabilizer decomposition of ρ provides a convenient way to calculate the eigenvalues of ρ. Since the stabilizers mutually commute, and the spectrum of each term in the decomposition is simply given by $\{+1, -1\}$, one finds the following expression for the eigenvalues:

$$\lambda_{j_1\ldots j_n} = \frac{1}{2^n} \sum_{i_1,\ldots,i_n=0}^{1} (-1)^{\sum_l i_l \cdot j_l} c_{i_1\ldots i_n} . \tag{5}$$

2.2. *Lowest purity compatible with measurement data*

In this section we determine the lowest purity compatible with measurement data in cluster state experiments. So we calculate the exact solution to the optimization problem (2). As seen in the previous section, it is legitimate to restrict to stabilizer diagonal states of the form (3). Then, the problem boils down to solving the following positive quadratic program:

$$\textit{minimize} \qquad P(c) = \frac{1}{2^n} \sum_{i_1,\ldots,\imath_n=0}^{1} c_{i_1\ldots i_n}^2 \tag{6}$$

$$\textit{s.t.} \qquad c_{\tilde{j}} = a_j \tag{7}$$

$$\lambda_{j_1\ldots j_n} \geq 0 \tag{8}$$

Clearly, this problem can be solved numerically by convex and quadratic optimization solvers such as SDTP, Sedumi, or Yalmip. However, the number of coefficients $c_{i_1\ldots i_n}$ grows exponentially with the number of qubits. Therefore, a numerical approach becomes practically intractable even for moderate qubit numbers. We will therefore derive an analytical solution.

Without loss of generality, we restrict the measurement values a_k to be positive. Any other choice could be accounted for by applying (local) unitaries to the density matrix, leaving the eigenvalues and thus the optimization problem unaffected.

Furthermore, we will only consider the case, in which the measurement outcomes of neighboring qubits fulfil $a_{k_1} + a_{k_2} \geq 1$. In other words, we restrict to the case where a non-zero fidelity with the desired stabilizer state can be guaranteed from the mean values a_k. This follows from the fact that the minimal fidelity one can infer from stabilizer measurements a_k is given by $F_{min} = \frac{\sum_{k=1}^{n} a_k - n + 2}{2}$ as shown in Ref. 10.

Now one can easily check that the choice

$$c_{i_1 \ldots i_n} = \sum_k i_k a_k - \sum_k i_k + 1 \tag{9}$$

fulfils all the constraints in the above quadratic program. Positivity of the eigenvalues follows from the fact that ρ is diagonal in the stabilizer basis. Thus, these coefficients represent a solution, which we denote by c^* in the following, to the primal problem.

Since the primal problem is convex, we may simply check the *Karush-Kuhn-Tucker* (KKT) conditions for optimality of the solution[16]. To be explicit, the KKT conditions require

$$\lambda_{j_1 \ldots j_n}(c^*) \geq 0, \tag{10}$$

$$g(c_{\tilde{j}} = c^*_{\tilde{j}}) := c_{\tilde{j}} - a_j = 0, \tag{11}$$

$$\mu_{j_1 \ldots j_n} \geq 0, \tag{12}$$

$$\mu_{j_1 \ldots j_n} \lambda_{j_1 \ldots j_n}(c^*) = 0, \tag{13}$$

$$\nabla_{j_1 \ldots j_n} P(c^*) - \sum_{i_1, \ldots, i_n = 0}^{1} \mu_{i_1 \ldots i_n} \nabla_{j_1 \ldots j_n} \lambda_{i_1 \ldots i_n} + \sum_{\tilde{m}=1}^{n} \nu_{\tilde{m}} \nabla_{j_1 \ldots j_n} g(c_{\tilde{m}}) = 0. \tag{14}$$

The first two conditions state that c^* is primal feasible, which is already proved. The last condition guarantees the optimality of c^*. Instead of deriving the dual quadratic program, we will now prove that Lagrange multipliers $\mu_{j_1 \ldots j_n}$ and $\nu_{\tilde{m}}$ can always be found to the solution c^*, such that the KKT conditions are fulfilled, thus showing that c^* is indeed optimal.

A valid solution for the Lagrange multipliers is provided in the following way: in order to satisfy (13) choose $\mu_{j_1 \ldots j_n} = 0$, if $\lambda_{j_1 \ldots j_n}(c^*) > 0$. Condition (14) can be rewritten as

$$\frac{2}{2^n} c - \frac{1}{2^n} A\mu + \nu = 0, \tag{15}$$

where it follows from Eq. 5 that A is the normalized Hadamard matrix with elements $A_{i_1, \ldots, i_n, j_1, \ldots, j_n} = (-1)^{\sum_k i_k j_k}$, $c = (c_{j_1 \ldots j_n})$, $\lambda = (\lambda_{j_1 \ldots j_n})$, and ν is the vector containing $\nu_{\tilde{m}}$ for $\tilde{m} = 1, \ldots, n$, while all other entries $\nu_{j_1 \ldots j_n} = 0$. Here the matrix A is chosen such that $Ac = 2^n \lambda$. Because of the orthogonality relation of the Hadamard matrix, and of $A = A^T$, it follows with $A^T A = 2^n I$:

$$\mu = A^T \left(\frac{2}{2^n} c + \nu\right) = 2\lambda + A\nu. \tag{16}$$

If $\lambda_{\vec{j}} > 0$ ($\vec{j} = (j_1, \ldots, j_n)$), then it must hold that $\mu_{\vec{j}} = 0$. Especially, in any case $\mu_{\vec{0}} = 0$, since the largest eigenvalue of ρ is $\lambda_{\vec{0}}$. Therefore, we have:

$$\sum_{\vec{m}\in I} (-1)^{\vec{j}\cdot\vec{m}} \nu_{\vec{m}} = -2\lambda_{\vec{j}}. \tag{17}$$

Here, I denotes the set of bit-strings with at most one bit unequal zero, as in the vector ν only the first n entries are unequal zero. More formally, $I = \{(0,\ldots,0),(1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$. This system of equations has the solution

$$\nu_{\vec{l}} = \lambda_{\vec{l}} - \lambda_{\vec{0}},\ \vec{l} \neq \vec{0}, \tag{18}$$

$$\nu_{\vec{0}} = -2\lambda_{\vec{0}} - \sum_{\vec{l}\neq\vec{0}} \nu_{\vec{l}}. \tag{19}$$

It remains to show that $\mu_{\vec{j}} \geq 0$ for $\lambda_{\vec{j}} = 0$:

$$\begin{aligned}
\mu_{\vec{j}} &= \sum_{\vec{m}} (-1)^{\vec{j}\cdot\vec{m}} \nu_{\vec{m}} = \nu_0 + \sum_{\vec{m}\neq\vec{0}} (-1)^{\vec{j}\cdot\vec{m}} \nu_{\vec{m}} && (20)\\
&= -2\lambda_{\vec{0}} - \sum_{\vec{l}\neq 0} \nu_l + \sum_{\vec{m}\neq 0} (-1)^{\vec{j}\cdot\vec{m}} \nu_{\vec{m}} && (21)\\
&= -2\lambda_{\vec{0}} + \sum_{\vec{m}\neq 0} ((-1)^{\vec{j}\cdot\vec{m}} - 1)\nu_{\vec{m}} && (22)\\
&= -2\lambda_{\vec{0}} + \sum_{\vec{m}\neq 0} ((-1)^{\vec{j}\cdot\vec{m}} - 1)(\lambda_{\vec{m}} - \lambda_{\vec{0}}) && (23)\\
&= -2\lambda_{\vec{0}} + \frac{1}{2^n} \sum_{\vec{m}\neq 0} ((-1)^{\vec{j}\cdot\vec{m}} - 1) \sum_{\vec{i}} ((-1)^{\vec{i}\cdot\vec{m}} - 1) c_{\vec{i}} && (24)\\
&= -2\lambda_{\vec{0}} + \frac{1}{2^n} \sum_{\vec{m}\neq 0} (-2\delta_{1,\vec{j}\cdot\vec{m}}) \sum_{\vec{i}} (-2\delta_{1,\vec{i}\cdot\vec{m}}) c_{\vec{i}}. && (25)
\end{aligned}$$

Since it is clear that $\lambda_{\vec{0}} > 0$, we can restrict our attention to eigenvalues $\lambda_{\vec{j}} = 0$ and dual variables $\mu_{\vec{j}}$ with at least one index $j_\alpha = 1$. Therefore, we have

$$\sum_{\vec{m}\neq 0} (-2\delta_{1,\vec{j}\cdot\vec{m}}) \sum_{\vec{i}} (-2\delta_{1,\vec{i}\cdot\vec{m}}) \frac{1}{2^n} c_{\vec{i}} \geq 4 \sum_{i_\beta=0,\beta\in\{1,\ldots,n\}-\alpha}^{1} \frac{1}{2^n} c_{i_1\ldots i_{\alpha-1}1i_{\alpha+1}\ldots i_n} \tag{26}$$

The RHS of the last inequality can be rewritten as

$$\frac{4}{2^n} \sum_{i_\beta=0,\beta\in\{1,\ldots,n\}-\alpha}^{1} c_{i_1\ldots i_{\alpha-1}1i_{\alpha+1}\ldots i_n} = 4(\lambda_{\vec{0}} - \sum_{i_\beta=0,\beta\in\{1,\ldots,n\}-\alpha}^{1} \frac{1}{2^n} c_{i_1\ldots i_{\alpha-1}0i_{\alpha+1}\ldots i_n}). \tag{27}$$

Employing Eq. (9), the last term may be evaluated as

$$\frac{1}{2^n}\sum_{i_\beta=0,\beta\in\{1,\dots,n\}-\alpha}^{1} c_{i_1\dots i_{\alpha-1}0i_{\alpha+1}\dots i_n} = \sum_{i_\beta=0,\beta\in\{1,\dots,n\}-\alpha}^{1}\frac{1}{2^n}(\sum_{k=1}^{n} i_k a_k - \sum_k i_k + 1) \tag{28}$$

$$= \frac{1}{4}(\sum_{k=2}^{n} a_k - n + 2) \leq \frac{\lambda_{\vec{0}}}{2}. \tag{29}$$

The last inequality follows from the fact that $\lambda_{\vec{0}} = \frac{1}{2}(\sum_{k=1}^{n} a_k - n + 2)$. Applying this result to Eq. (25) immediately implies $\mu_{\vec{j}} \geq 0$, if $\lambda_{\vec{j}} = 0$. Therefore, all KKT conditions are fulfilled, proving that the solution c^* is indeed optimal.

3. Example: Two qubit purity

For the purpose of illustration, let us consider the simple example of two qubits, supposedly prepared as a cluster state. Given only the outcomes of the stabilizer measurements $a_1 = tr(\rho X \otimes Z)$ and $a_2 = tr(\rho Z \otimes X)$, we seek the lowest purity compatible with these two measurements. Using the symmetries of the stabilizers, we may restrict the problem to density matrices of the form

$$\rho = \frac{1}{4}(c_{00}\mathbf{1} + c_{10}K_1 + c_{01}K_2 + c_{11}K_1K_2). \tag{30}$$

Because of $tr(\rho) = 1$, $c_{00} = 1$. Furthermore, the measurements determine $c_{10} = a_1$ and $c_{01} = a_2$. In order to find the minimal purity, the objective is now to minimize

$$P_{min} = \frac{1}{4}(c_{00}^2 + c_{10}^2 + c_{01}^2 + c_{11}^2) \tag{31}$$

subject to $\lambda_{j_1 j_2}(c) = \sum_{i_1,i_2=0}^{1}(-1)^{i_1 j_1 + i_2 j_2} c_{i_1 i_2} \geq 0$.

As solution to the primal problem, one chooses $c_{11} = \frac{1}{4}(a_1 + a_2 - 1)$. Then the eigenvalues of ρ are simply given by

$$\lambda_{00} = \frac{1}{2}(a_1 + a_2) \tag{32}$$

$$\lambda_{10} = \frac{1}{2}(1 - a_1) \tag{33}$$

$$\lambda_{01} = \frac{1}{2}(1 - a_2) \tag{34}$$

$$\lambda_{11} = 0 \tag{35}$$

thus fulfilling the positivity constraint. To see the optimality of the solution, one can check the remaining KKT conditions.

$$\frac{1}{2}-\mu_1-\mu_2-\mu_3-\mu_4+\nu_1 = 0, \tag{36}$$

$$\frac{a_1}{2}-\mu_1+\mu_2-\mu_3+\mu_4+\nu_2 = 0, \tag{37}$$

$$\frac{a_2}{2}-\mu_1-\mu_2+\mu_3+\mu_4+\nu_3 = 0, \tag{38}$$

$$\frac{1}{2}(a_1+a_2-1)-\mu_1+\mu_2+\mu_3-\mu_4 = 0 \tag{39}$$

The Lagrange multipliers ν are determined by

$$\nu_{10}=\lambda_{10}-\lambda_{00}=\frac{1}{2}(1-2a_1-a_2), \tag{40}$$

$$\nu_{01}=\lambda_{01}-\lambda_{00}=\frac{1}{2}(1-2a_2-a_1). \tag{41}$$

$$\nu_{00}=-2\lambda_{00}-\nu_{10}-\nu_{01}=\frac{1}{2}(a_1+a_2)-1. \tag{42}$$

If the measurement outcomes fulfil $a_1, a_2 > 0$, which should be the case in a cluster state experiment, then $\lambda_{00}, \lambda_{10}, \lambda_{01} > 0$, and $\lambda_{11} = 0$. Therefore, $\mu_1 = \mu_2 = \mu_3 = 0$, to fulfill the KKT condition (13). It is now straightforward to check that the choice of these Lagrange multipliers satisfy conditions (36) through (39), thus proving optimality of the solution.

4. Quality of the bounds

In this section we will test the usefulness of the obtained bound on the purity. For this purpose, let us consider a perfect cluster state, which is subject to dephasing for a certain time t. The system is then described by a master equation of the form

$$\dot{\rho}=\frac{\gamma}{2}\sum_i(Z_i\rho Z_i-\rho), \tag{43}$$

where γ is the dephasing rate. As an example, the dephasing constants are chosen here such that $\gamma t = 0.1$. The results given in Tab. 1 show that the estimated purity differs only by a few per cent from the exact value despite the simple form of the analytical bound.

Let us now consider locally-dephased GHZ states, which are obtained via the same master equation as in the cluster state case. The stabilizer operators of a GHZ state are $K_1 = X \otimes ... \otimes X$ and $K_k = Z_{k-1}Z_k$, $k = 2, ..., n$. In Tab. 2 the exact purities of noisy GHZ states are compared with the estimate from stabilizer measurements. It should not be too surprising that the estimate gives the exact purity if the qubits are subject to local dephasing. The reason is that the dephasing only reduces the non-diagonal elements of the density matrix, thus the mean values of the stabilizers a_k, $k \geq 2$, remain 1. So the decoherence is due to the reduced "corner elements" of the density matrix, which are measured by a_1. In this way, the noise does not lead to an information loss.

Table 1. Comparison between exact and estimated purities for noisy cluster states

No. qubits	exact purity	estimated purity	relative deviation
2	0.8269	0.8233	0.0044
3	0.7520	0.7417	0.0137
4	0.6838	0.6646	0.0281

Table 2. Comparison between exact and estimated purities for noisy GHZ states

No. qubits	exact purity	estimated purity	relative deviation
2	0.8352	0.8352	0.0000
3	0.7744	0.7744	0.0000
4	0.7247	0.7247	0.0000

5. Estimation with error bars

Needless to say, due to a finite number of measurements and experimental imperfections, the measurement outcomes a_i we considered in previous sections posses an error $\Delta a_i \geq 0$, which clearly affects the result of a purity estimation. In this section we briefly address the question how to take into account such errors.

Let us assume we measured the stabilizers K_i with measurement outcomes $a_i \pm \Delta a_i = tr(\rho K_i)$. In order to permit a semidefinite programming approach, we then formulate the problem as

$$P_{min} = min\{tr[\rho^2] : a_i + \Delta a_i \geq tr(K_i \rho), tr(K_i \rho) \geq a_i - \Delta a_i, \rho \geq 0\}. \qquad (44)$$

As seen in Sec. 2.2, the optimal solution to the purity minimization problem (2) is given by a stabilizer diagonal representation of the density matrix with coefficients $c_{i_1 \dots i_n} = \sum_k i_k a_k - \sum_k i_k + 1$. From this, it can be easily seen that the choice $c^{(\pm)}_{i_1 \dots i_n} = \sum_k i_k (a_k \pm \Delta a_k) - \sum_k i_k + 1$ leads to upper (lower) error estimates of the minimal purity with objective values $P^{(\pm)}_{min} = \frac{1}{2^n} \sum^{1}_{i_1,\dots,i_n=0} (c^{(\pm)}_{i_1 \dots i_n})^2$.

In this way, we obtain an analytical result on the minimal purity taking into account errors in the stabilizer measurements. It might be that other methods such as the Kalman-filter approach developed by Audenaert and Scheel [18] might be employed here. However, this is unlikely to deliver analytical results, and it is beyond the scope of this paper.

6. Entropy estimation

Alongside the purity of a quantum state $tr(\rho^2)$, the entropy is another quantifier for the degree of disorder of a system. Given only the measurement outcomes of the stabilizer, one can employ similar techniques as in the case of the purity to estimate the entropy.

The von Neumann entropy is given by $S = -tr(\rho log \rho)$. Again, let us consider the generators of the stabilizer group as our observables with measurement outcomes $a_1 = tr(\rho K_1), \dots, a_n = tr(\rho K_n)$. Then we may restrict to states diagonal in the stabilizer basis $\rho = \frac{1}{2^n}\sum c_{i_1\dots i_n} K_1^{i_1}\dots K_n^{i_n}$ with eigenvalues $\lambda_{j_1\dots j_n}(\rho) = \sum_{i_1,\dots,i_n=0}^{1}(-1)^{\sum_k i_k j_k} c_{i_1\dots i_n}$. Now, we need to maximize the entropy over all states predicting the measurement outcomes a_k, $k \in 1,\dots,n$.

$$S_{max} = max\{S(\rho) : tr(\rho K_i) = a_i, \rho \geq 0\} \tag{45}$$

$$= max\{-\sum_{i_1,\dots,i_n=0}^{1} \lambda_{i_1\dots i_n} log \lambda_{i_1\dots i_n} : \lambda_{i_1\dots i_n} \geq 0, c_{\vec{j}} = a_j\} \tag{46}$$

Such an entropy maximization is known as Jaynes principle in statistical physics and can be solved using the partition function technique. Then the reconstructed density matrix ρ_J is given by the prescription: $\rho_J = Z^{-1}\exp(-\sum_n \eta_n K_n)$ with the partition function $Z(\eta_1,\dots,\eta_n) = tr[\exp(-\sum_n \eta_n K_n)]$. From this one can then derive equations for the Lagrange multipliers $\eta_1,\dots,\eta_n$.

Here, we demonstrate a direct way to determine the exact maximum entropy state in the following way: the outcomes a_k of the stabilizer measurements give rise to the probabilities $p_k^{(\pm)} = \frac{1\pm a_k}{2}$ for the projections upon the stabilizer eigenspaces. Furthermore, we denote the probability distribution of the joint state of the system by $\lambda_{i_1\dots i_n}$. It is a well-known property of the entropy to be subadditive, that is, the total entropy is bounded by the sum of the entropies of the marginals

$$S(\lambda_{i_1\dots i_n}) \leq \sum_{k=1,s=\pm}^{n} S(p_k^{(s)}). \tag{47}$$

We will now prove that the above relation holds with equality for the probability distribution $\lambda_{i_1\dots i_n} = \prod_{k=1}^{n}\frac{1+(-1)^{i_k}a_k}{2}$, thus giving the exact maximal entropy $S_{max} = -\sum_{i_1,\dots,i_n=0}^{1}\lambda_{i_1\dots i_n} log \lambda_{i_1\dots i_n}$. Because the distribution $\lambda_{i_1\dots i_n}$ is a product of probabilities, the following holds:

$$-S(\lambda_{\vec{i}}) = \sum_{\vec{i}} \lambda_{\vec{i}} log(\lambda_{\vec{i}}) = \sum_{\vec{i}}(\prod_l \frac{1+(-1)^{i_l}a_l}{2}) log(\prod_k \frac{1+(-1)^{i_k}a_k}{2}) \tag{48}$$

$$= \sum_{\vec{i}}(\prod_l \frac{1+(-1)^{i_l}a_l}{2}) \sum_k log\frac{1+(-1)^{i_k}a_k}{2} \tag{49}$$

$$= \sum_k \sum_{\vec{i}}(\prod_{l\neq k} \frac{1+(-1)^{i_l}a_l}{2}) \frac{1+(-1)^{i_k}a_k}{2} log\frac{1+(-1)^{i_k}a_k}{2} \tag{50}$$

$$= \sum_k \sum_{i_k=0}^{1} \frac{1+(-1)^{i_k}a_k}{2} log\frac{1+(-1)^{i_k}a_k}{2} = -\sum_{k,s=\pm} S(p_k^{(s)}) \tag{51}$$

Using the eigenvalue equation (5), one can immediately find the quantum state giving this distribution:

$$\rho = \frac{1}{2^n}\sum_{i_1,\dots,i_n=0}^{1}(a_1K_1)^{i_1}\dots(a_nK_n)^{i_n}. \tag{52}$$

One might ask how close the minimum purity state is to the maximum entropy state. To give an intuiton to this question, we consider initially perfect cluster states subject to dephasing according to Eq. (43). One obtains the results given in the table 3 showing that the entropy of the minimum purity state differs only slightly from the maximum entropy.

Table 3. Comparison between exact maximum entropy and estimated entropies from the minimum purity state for noisy cluster states

No. qubits	maximal entropy	entropy of minimum purity state	relative deviation
2	0.3827	0.3803	0.0063
3	0.5740	0.5667	0.0127
4	0.7653	0.7505	0.0193

7. Conclusion

In this paper we have developed techniques to estimate purities and entropies in stabilizer state experiments with few measurements. We assume only that the generators of the stabilizer group are measured in an experiment. Therefore, the number of local measurements is linear in the qubit number, compared to an exponential growth in the number of measurement settings for full-state tomography. Given only these measurement outcomes, we have developed the optimal analytical solution to the minimal purity compatible with the measurement data. A comparison of this bound on the purity with the exact purity calculated from simulated data of noisy cluster states shows that, despite the little number of measurements, the bound on the purity differs only by a few per cent from the exact value. Furthermore, we have derived a direct and exact analytical solution to the maximum entropy that can be inferred from stabilizer measurements.

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